Chapter 13

Radiation

13-1 Liénard-Wiechert Potentials

A. RETARDED POTENTIAL

(1) Maxwell's equations

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \cdots\cdots \text{Faraday's law}$$
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \cdots\cdots \text{Gauss's law}$$
$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \cdots\cdots \text{Ampère's law and displacement current}$$
$$\nabla\cdot\vec{B} = 0$$

(2) In *static* cases,

$$\frac{\partial\vec{E}}{\partial t} = 0, \qquad \frac{\partial\vec{B}}{\partial t} = 0$$

Maxwell's equations becomes

$$\nabla\times\vec{E} = 0$$
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \cdots\cdots \text{Gauss's law}$$
$$\nabla\times\vec{B} = \mu_0\vec{J} \cdots\cdots \text{Ampère's law}$$
$$\nabla\cdot\vec{B} = 0$$

The uniqueness of the electric field $\vec{E}(\vec{r})$

1. According to Helmholtz theorem:

$$\nabla\times\vec{E} = 0 \text{ and } \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \text{ with } \vec{E}(r\to\infty) \xrightarrow{\text{rapidly}} 0$$

we obtain

$$\vec{E} = -\nabla\varphi$$

where

$$\varphi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(r')}{|\vec{r}-\vec{r}'|}d^3r'$$

2. Solve the Poisson's equation with suitable boundary conditions and φ is uniquely determined.

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} \Rightarrow \vec{E} = -\nabla\varphi$$

The uniqueness of the magnetic field $\vec{B}(\vec{r})$

1. According to Helmholtz theorem:

$$\nabla \times \vec{B} = \mu_0 \vec{J} \text{ and } \nabla \cdot \vec{B} = 0 \text{ with } \vec{B}(r \to \infty) \xrightarrow{\text{rapidly}} 0$$

we obtain

$$\vec{B} = \nabla \times \vec{A}(\vec{r})$$

where

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r'$$

(3) In *electrodynamic* cases,

Consider the electric field $\vec{E}(\vec{r}, t)$:

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$$

$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \varphi$$

$$\Rightarrow \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}$$

Substituting into Gauss's law, we obtain

$$\nabla \cdot \left(-\nabla \varphi - \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \varphi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \cdots\cdots \text{(a)}$$

Consider the magnetic fields $\vec{B}(\vec{r}, t)$:

$$\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}(\vec{r})$$

Substituting into Ampère's law, we obtain

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla \varphi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\Rightarrow \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \nabla \frac{\partial \varphi}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\Rightarrow \left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \varphi}{\partial t}\right) = -\mu_0 \vec{J} \cdots\cdots \text{(b)}$$

(4) Since $\vec{E}$ and $\vec{B}$ are physical quantities, we are free to impose extra conditions on φ and $\vec{A}$ without changing $\vec{E}$ and $\vec{B}$, such as,

$$\vec{A}'(\vec{r}, t) = \vec{A} + \nabla \lambda(\vec{r}, t)$$

$$\varphi'(\vec{r}, t) = \varphi - \frac{\partial}{\partial t} \lambda(\vec{r}, t)$$

VERIFYING:

$$\vec{E} = -\nabla\varphi' - \frac{\partial \vec{A}'}{\partial t}$$
$$= -\nabla\left(\varphi - \frac{\partial \lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla\lambda\right)$$
$$= -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} + \nabla\frac{\partial \lambda}{\partial t} - \frac{\partial}{\partial t}\nabla\lambda$$
$$= -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \nabla \times \vec{A}'$$
$$= \nabla \times \left(\vec{A} + \nabla\lambda\right)$$
$$= \nabla \times \vec{A} + \underbrace{\nabla \times \nabla\lambda}_{=0}$$
$$= \nabla \times \vec{A}$$

Such changes in φ and $\vec{A}$ are called gauge transformation, and $\vec{E}$ and $\vec{B}$ are gauge invariant.

(5) Equations (a) and (b) can be simplified by choosing different conditions as

1. $\nabla \cdot \vec{A} = 0$ ····· called the Coulomb gauge
   We obtain
   $$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} \text{ ····· (a')}$$
   $$\left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\mu_0\epsilon_0\frac{\partial\varphi}{\partial t}\right) = -\mu_0\vec{J} \text{ ····· (b')}$$
2. $\nabla \cdot \vec{A} + \mu_0\epsilon_0\frac{\partial\varphi}{\partial t} = 0$ ····· called the Lorentz gauge
   We obtain
   $$\nabla^2\varphi + \frac{\partial}{\partial t}\left(-\mu_0\epsilon_0\frac{\partial\varphi}{\partial t}\right) = -\frac{\rho}{\epsilon_0}$$
   $$\Rightarrow \nabla^2\varphi - \mu_0\epsilon_0\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \text{ ····· (a')}$$
   $$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \text{ ····· (b')}$$

(6) We then can define an operator and obtain the symmetric form of the differential equations for $\varphi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$.

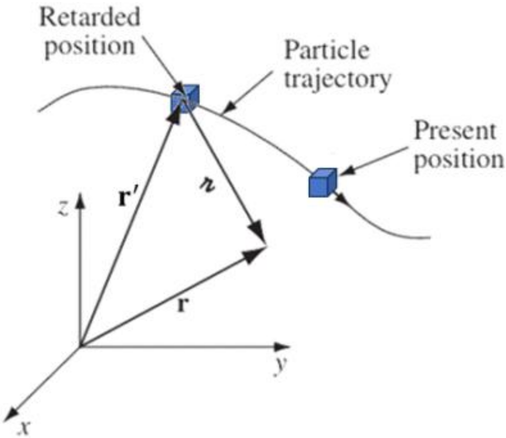
$$\hat{\Box}^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}$$

$$\Rightarrow \begin{cases} \hat{\Box}^2 \varphi = -\dfrac{\rho}{\epsilon_0} \\ \hat{\Box}^2 \vec{A} = -\mu_0 \vec{J} \end{cases}$$

where

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}',t)}{\mathscr{r}} d\tau' \text{ and } \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}',t)}{\mathscr{r}} d\tau'$$

(7) In electrodynamics, the potentials at the present time t depend upon the charge and current densities at times $\le t$. Thus, we should consider the status of the source at some earlier time t_r, called the retarded time.



$$t_r = t - \frac{\mathscr{r}}{c}$$

φ and $\vec{A}$ are obtained as

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}} d\tau' \text{ and } \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}} d\tau'$$

Because the integrands are evaluated at the retarded time, these are called retarded potentials.

EXAMPLES:

1. Verifying the retarded potential $\varphi(\vec{r},t)$ satifies the inhomogeneous equation $\hat{\Box}^2 \varphi = -\rho/\epsilon_0$.
   ANSWER:

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

Here

$$\nabla\rho = \dot{\rho}\nabla t_r = -\frac{\dot{\rho}}{c}\nabla\mathscr{r} = -\frac{\dot{\rho}}{c}\hat{\mathscr{r}}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

so

$$\nabla\varphi = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

We obtain

$$\nabla^2\varphi = \frac{1}{4\pi\epsilon_0}\int\left(-\frac{1}{c}\left[(\nabla\dot{\rho})\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} + \dot{\rho}\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right)\right] - \left[(\nabla\rho)\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \rho\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]\right)d\tau'$$

Here

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\hat{\mathscr{r}}$$

so

$$\nabla^2\varphi = \frac{1}{4\pi\epsilon_0}\int\left(-\frac{1}{c}\left[\left(-\frac{1}{c}\ddot{\rho}\hat{\mathscr{r}}\right)\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}} + \dot{\rho}\frac{1}{\mathscr{r}^2}\right] - \left[\left(-\frac{\dot{\rho}}{c}\hat{\mathscr{r}}\right)\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \rho 4\pi\delta^3(\mathscr{r})\right]\right)d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\mathscr{r})\right]d\tau'$$

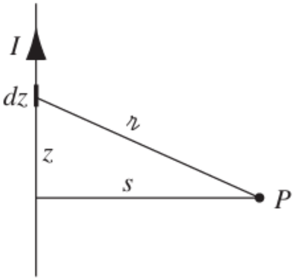
$$= \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\rho(\vec{r},t)}{\epsilon_0}$$

$$\Rightarrow \hat{\Box}^2\varphi = \nabla^2\varphi - \mu_0\epsilon_0\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\epsilon_0}$$

2. An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \le 0 \\ I_0, & \text{for } t > 0 \end{cases}$$

Find the resulting electric and magnetic fields.



ANSWER:

$$\vec{A}(s,t)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}dz\hat{z}$$

For $t<s/c$, the fields has not yet reached P, and the potential is zero.

For $t>s/c$, only the segment contributes

$$|z|\le\sqrt{(ct)^2-s^2}$$

Thus, we obtain

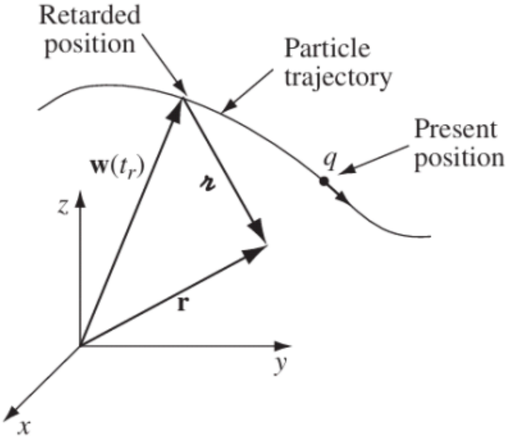
$$\vec{A}(s,t)=\frac{\mu_0}{4\pi}2\int_0^{\sqrt{(ct)^2-s^2}}\frac{I_0}{\sqrt{s^2+z^2}}dz\hat{z}$$
$$=\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{z}$$

The electric and magnetic fields are

$$\vec{E}(s,t)=-\frac{\partial\vec{A}}{\partial t}=-\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\hat{z}$$
$$\vec{B}(s,t)=\nabla\times\vec{A}$$
$$=-\frac{\partial A_z}{\partial s}\hat{\phi}$$
$$=\frac{\mu_0 I_0 ct}{2\pi s\sqrt{(ct)^2-s^2}}\hat{\phi}$$
$$=\frac{\mu_0 I_0}{2\pi s\sqrt{1-(s/ct)^2}}\hat{\phi}$$

B. RETARDED POTENTIAL OF A MOVING POINT CHARGE

(1) Consider a point charge moving on a specified trajectory



The scalar potential of a point charge

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t')}{\mathscr{r}}\delta(t'-t_r)\,dt'd\tau'$$

The charge density for a point charge moving along the path $\vec{w}(t)$ is given by

$$\rho(\vec{r},t) = q\delta^3(\vec{r}-\vec{w}(t))$$

Thus, we have

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q\delta^3(\vec{r}'-\vec{w}(t'))}{\mathscr{r}}\delta(t'-t_r)\,dt'd\tau'$$

$$= \frac{e}{4\pi\epsilon_0}\int \frac{1}{|\vec{r}-\vec{w}\,(t')|}\delta\left(t'-t+\frac{|\vec{r}-\vec{w}\,(t')|}{c}\right)dt'$$

Since

$$\delta(f(x)) = \delta(x-x_0)\left|\frac{\partial f}{\partial x}\right|^{-1}$$

we have

$$\delta\left(t'-t+\frac{|\vec{r}-\vec{w}\,(t')|}{c}\right) = \delta(t'-t_r)\left|\frac{\partial}{\partial t'}\left(t'-t+\frac{|\vec{r}-\vec{w}\,(t')|}{c}\right)\right|^{-1}$$

Here

$$\frac{\partial}{\partial t'}\left(t' - t + \frac{\left|\vec{r} - \vec{w}\,(t')\right|}{c}\right) = 1 + \frac{1}{c}\frac{\partial}{\partial t'}\Big(\big(\vec{r} - \vec{w}\,(t')\big)\big(\vec{r} - \vec{w}\,(t')\big)\Big)^{1/2}$$
$$= 1 + \frac{1}{c}\Big(\big(\vec{r} - \vec{w}\,(t')\big)\big(\vec{r} - \vec{w}\,(t')\big)\Big)^{-1/2}$$
$$\cdot \big(\vec{r} - \vec{w}\,(t')\big) \cdot \frac{\partial}{\partial t'}\big(-\vec{w}\,(t')\big)$$
$$= 1 - \frac{1}{c}\frac{\big(\vec{r} - \vec{w}\,(t')\big) \cdot \frac{\partial}{\partial t'}\big(\vec{w}\,(t')\big)}{\left|\vec{r} - \vec{w}\,(t')\right|}$$
$$= 1 - \frac{\vec{v}}{c} \cdot \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$
$$= 1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}$$

Thus, we obtain

$$\delta\left(t' - t + \frac{\left|\vec{r} - \vec{w}\,(t')\right|}{c}\right) = \delta\big(t' - t_r\big)\left|1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right|^{-1}$$
$$= \delta\big(t' - t_r\big)\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^{-1}$$

Therefore, the retarded scalar potential for a moving point charge is

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\int \frac{1}{\left|\vec{r} - \vec{w}\,(t')\right|}\delta\big(t' - t_r\big)\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^{-1} dt'$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^{-1}, \text{ where } \mathscr{r} = \left|\vec{r} - \vec{w}\,(t_r)\right|$$
$$= \frac{q}{4\pi\epsilon_0 \mathscr{r}\big(1 - \vec{v} \cdot \hat{\mathscr{r}}/c\big)}$$

(2) Moreover, since the current density is $\rho\vec{v}$, the retarded vector potential is

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\rho\big(\vec{r}', t_r\big)\vec{v}\big(t_r\big)}{\mathscr{r}} d\tau'$$
$$= \frac{\mu_0 \vec{v}\big(t_r\big)}{4\pi}\int \frac{\rho\big(\vec{r}', t_r\big)}{\mathscr{r}} d\tau'$$
$$= \frac{\mu_0}{4\pi}\frac{\vec{v}\big(t_r\big)}{\mathscr{r}\big(1 - \vec{v} \cdot \hat{\mathscr{r}}/c\big)}$$

$\varphi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ are the Liénard-Wiechert Potential for a moving point charge.

EXAMPLES:

1. Find the potentials of a point charge moving with constant velocity.

   ANSWER:

$$\vec{w}(t) = \vec{v}t$$

The retarded time is

$$t_r = t - \frac{\mathscr{r}}{c} = t - \frac{|\vec{r} - \vec{v}t_r|}{c}$$

$$\Rightarrow |\vec{r} - \vec{v}t_r| = c(t - t_r)$$

$$\Rightarrow r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

$$\Rightarrow t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Suppose the charge is at rest at the origin ($v = 0$),

$$t_r = \frac{(c^2t) \pm \sqrt{(c^2t)^2 + (c^2)(r^2 - c^2t^2)}}{c^2} = t \pm \frac{r}{c}$$

the retarded time should be

$$t_r = t - \frac{r}{c}$$

Thus, we obtain

$$t_r = \frac{(c^2t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Therefore, we can calculate

$$\mathscr{r}(1 - \vec{v}\cdot\hat{\mathscr{r}}/c) = c(t - t_r)\left[1 - \frac{\vec{v}}{c}\cdot\frac{(\vec{r} - \vec{v}t_r)}{c(t - t_r)}\right]$$

$$= c(t - t_r) - \frac{\vec{v}}{c}\cdot(\vec{r} - \vec{v}t_r)$$

$$= \frac{1}{c}\left[(c^2t - \vec{v}\cdot\vec{r}) - (c^2 - v^2)t_r\right]$$

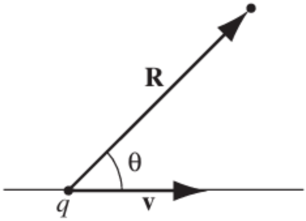
$$= \frac{1}{c}\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}$$

The retarded potentials are

$$\varphi(\vec{r}, t) = \frac{qc}{4\pi\epsilon_0\sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\sqrt{\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+(c^2-v^2)(r^2-c^2t^2)}}$$

2. Let $\vec{R}=\vec{r}-\vec{v}t$ be the vector from the present position of the particle to the field point $\vec{r}$, and θ is the angle between $\vec{R}$ and $\vec{v}$.



Find the scalar retarded potential for a point charge moving with constant velocity.

ANSWER:

$$\varphi(\vec{r},t)=\frac{qc}{4\pi\epsilon_0\sqrt{\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+(c^2-v^2)(r^2-c^2t^2)}}$$

Here

$$\left(c^2t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)$$
$$=c^4t^2-2c^2t(\vec{r}\cdot\vec{v})+\left(\vec{r}\cdot\vec{v}\right)^2+c^2r^2-c^4t^2-v^2r^2+c^2v^2t^2$$
$$=-c^22(\vec{r}\cdot\vec{v}t)+\left(\vec{r}\cdot\vec{v}\right)^2+c^2r^2-v^2r^2+c^2v^2t^2$$

Since

$$\vec{R}=\vec{r}-\vec{v}t$$
$$\Rightarrow R^2=r^2-2(\vec{r}\cdot\vec{v}t)+v^2t^2$$
$$\Rightarrow 2(\vec{r}\cdot\vec{v}t)=r^2+v^2t^2-R^2$$

we obtain

$$
\begin{aligned}
&\left(c^2t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2t^2\right) \\
&\quad = -c^2\left(r^2 + v^2t^2 - R^2\right) + \left(\vec{r}\cdot\vec{v}\right)^2 + c^2r^2 - v^2r^2 + c^2v^2t^2 \\
&\quad = \left(\vec{r}\cdot\vec{v}\right)^2 - v^2r^2 + c^2R^2 \\
&\quad = \left(\left(\vec{R} + \vec{v}t\right)\cdot\vec{v}\right)^2 - \left(\vec{R} + \vec{v}t\right)^2 v^2 + c^2R^2 \\
&\quad = \left(\vec{R}\cdot\vec{v}\right)^2 + 2\left(\vec{R}\cdot\vec{v}\right)v^2t + v^4t^2 \\
&\qquad\qquad - R^2v^2 - 2\left(\vec{R}\cdot\vec{v}\right)v^2t - v^4t^2 + c^2R^2 \\
&\quad = \left(\vec{R}\cdot\vec{v}\right)^2 - R^2v^2 + c^2R^2 \\
&\quad = R^2v^2\cos^2\theta - R^2v^2 + c^2R^2 \\
&\quad = c^2R^2 - R^2v^2\left(1 - \cos^2\theta\right) \\
&\quad = c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\varphi\left(\vec{r}, t\right) &= \frac{qc}{4\pi\epsilon_0\sqrt{c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)}} \\
&= \frac{q}{4\pi\epsilon_0 R\sqrt{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)}}
\end{aligned}
$$

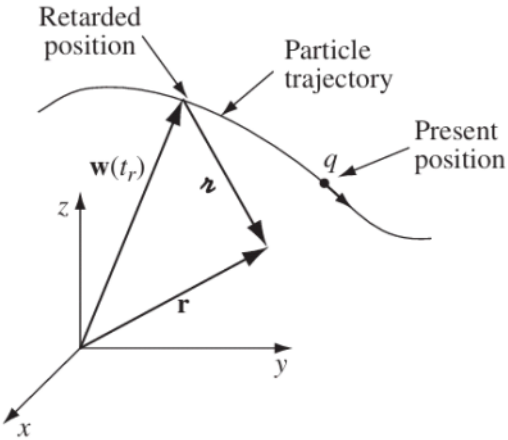
where

$$
\theta = \cos^{-1}\left(\vec{R}\cdot\vec{v}\right)
$$

13-2 The Fields of a Moving Point Charge

A. THE FIELDS OF A MOVING POINT CHARGE

(1) The fields of a point charge in arbitrary motion are



$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A}$$

where φ and $\vec{A}$ are the Liénard-Wiechert potentials,

$$\varphi(\vec{r},t) = \frac{q}{4\pi\epsilon_0 \mathscr{r}(1 - \vec{v}\cdot\hat{\mathscr{r}}/c)}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\vec{v}(t_r)}{\mathscr{r}(1 - \vec{v}\cdot\hat{\mathscr{r}}/c)} = \frac{\vec{v}(t_r)}{c^2}\varphi(\vec{r},t)$$

and

$$\mathscr{r} = |\vec{r} - \vec{w}\,(t_r)|$$

$$\vec{v} = \dot{\vec{w}}(t_r)$$

$$t_r = t - \frac{|\vec{r} - \vec{w}\,(t_r)|}{c} \Rightarrow \mathscr{r} = c(t - t_r)$$

(2) The gradient of $\varphi(\vec{r},t)$ is

$$\nabla\varphi = \frac{q}{4\pi\epsilon_0}\frac{-1}{\left(\mathscr{r}(1 - \vec{v}\cdot\hat{\mathscr{r}}/c)\right)^2}\nabla\left(\mathscr{r}\left(1 - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c}\right)\right)$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{\nabla\mathscr{r} - \frac{1}{c}\nabla(\vec{v}\cdot\vec{\mathscr{r}})}{\left(\mathscr{r}(1 - \vec{v}\cdot\hat{\mathscr{r}}/c)\right)^2}$$

where

$$\nabla \mathscr{r} = -c\nabla t_r$$

$$\nabla(\vec{v}\cdot\vec{\mathscr{r}}) = \underbrace{(\vec{\mathscr{r}}\cdot\nabla)\vec{v}}_{①} + \underbrace{(\vec{v}\cdot\nabla)\vec{\mathscr{r}}}_{②} + \underbrace{\vec{\mathscr{r}}\times(\nabla\times\vec{v})}_{③} + \underbrace{\vec{v}\times(\nabla\times\vec{\mathscr{r}})}_{④}$$

OS:

$$\vec{\mathscr{r}}\times(\nabla\times\vec{v}) = \frac{1}{2}\nabla(\vec{v}\cdot\vec{\mathscr{r}}) - (\vec{\mathscr{r}}\cdot\nabla)\vec{v}$$

$$\vec{v}\times(\nabla\times\vec{\mathscr{r}}) = \frac{1}{2}\nabla(\vec{v}\cdot\vec{\mathscr{r}}) - (\vec{v}\cdot\nabla)\vec{\mathscr{r}}$$

① For the $(\vec{\mathscr{r}}\cdot\nabla)\vec{v}$ term:

$$\begin{aligned}(\vec{\mathscr{r}}\cdot\nabla)\vec{v} &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\vec{v}(t_r) \\ &= z\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x} + y\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial y} + z\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial z} \\ &= \left(x\frac{\partial t_r}{\partial x} + y\frac{\partial t_r}{\partial y} + z\frac{\partial t_r}{\partial z}\right)\frac{d\vec{v}}{dt_r} \\ &= (\vec{\mathscr{r}}\cdot\nabla t_r)\vec{a}\end{aligned}$$

② For the $(\vec{v}\cdot\nabla)\vec{\mathscr{r}}$ term:

$$(\vec{v}\cdot\nabla)\vec{\mathscr{r}} = (\vec{v}\cdot\nabla)\vec{r} - (\vec{v}\cdot\nabla)\vec{w}(t_r)$$

$$\begin{aligned}(\vec{v}\cdot\nabla)\vec{r} &= \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z}) \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \\ &= \vec{v}\end{aligned}$$

$$(\vec{v}\cdot\nabla)\vec{w}(t_r) = (\vec{v}\cdot\nabla t_r)\vec{v}$$

$$\Rightarrow (\vec{v}\cdot\nabla)\vec{\mathscr{r}} = \vec{v} - (\vec{v}\cdot\nabla t_r)\vec{v}$$

③ For the $\vec{\mathscr{r}}\times(\nabla\times\vec{v})$ term:

$$\left(\nabla\times\vec{v}\right)=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$
$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial t_r}{\partial x}\frac{d}{dt_r} & \frac{\partial t_r}{\partial y}\frac{d}{dt_r} & \frac{\partial t_r}{\partial z}\frac{d}{dt_r}\\ v_x & v_y & v_z\end{vmatrix}$$
$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial t_r}{\partial x} & \frac{\partial t_r}{\partial y} & \frac{\partial t_r}{\partial z}\\ \frac{dv_x}{dt_r} & \frac{dv_y}{dt_r} & \frac{dv_z}{dt_r}\end{vmatrix}$$
$$=-\vec{a}\times\nabla t_r$$
$$\Rightarrow \vec{\mathscr{r}}\times\left(\nabla\times\vec{v}\right)=\underbrace{\vec{\mathscr{r}}\times\left(-\vec{a}\times\nabla t_r\right)}_{\text{BAC-CAB rule}}=-\vec{a}\left(\vec{\mathscr{r}}\cdot\nabla t_r\right)+\left(\vec{\mathscr{r}}\cdot\vec{a}\right)\nabla t_r$$

④ For the $\vec{v}\times\left(\nabla\times\vec{\mathscr{r}}\right)$ term:

$$\left(\nabla\times\vec{\mathscr{r}}\right)=\underbrace{\nabla\times\vec{r}}_{=0}-\nabla\times\vec{w}=\vec{v}\times\nabla t_r$$
$$\Rightarrow \vec{v}\times\left(\nabla\times\vec{\mathscr{r}}\right)=\underbrace{\vec{v}\times\left(\vec{v}\times\nabla t_r\right)}_{\text{BAC-CAB rule}}=\vec{v}\left(\vec{v}\cdot\nabla t_r\right)-v^2\nabla t_r$$

Finally, collecting all terms, we obtain

$$\nabla\left(\vec{v}\cdot\vec{\mathscr{r}}\right)=\underbrace{\left(\vec{\mathscr{r}}\cdot\nabla t_r\right)\vec{a}}_{①}+\underbrace{\vec{v}-\left(\vec{v}\cdot\nabla t_r\right)\vec{v}}_{②}$$
$$\underbrace{-\vec{a}\left(\vec{\mathscr{r}}\cdot\nabla t_r\right)+\left(\vec{\mathscr{r}}\cdot\vec{a}\right)\nabla t_r}_{③}+\underbrace{\vec{v}\left(\vec{v}\cdot\nabla t_r\right)-v^2\nabla t_r}_{④}$$
$$=\vec{v}+\left(\vec{\mathscr{r}}\cdot\vec{a}\right)\nabla t_r-v^2\nabla t_r$$
$$=\vec{v}+\left(\left(\vec{\mathscr{r}}\cdot\vec{a}\right)-v^2\right)\nabla t_r$$

The gradient of $\varphi\left(\vec{r},t\right)$ becomes

$$\nabla\varphi=-\frac{q}{4\pi\epsilon_0}\frac{-c\nabla t_r-\frac{1}{c}\left(\vec{v}+\left(\left(\vec{\mathscr{r}}\cdot\vec{a}\right)-v^2\right)\nabla t_r\right)}{\left(\mathscr{r}\left(1-\vec{v}\cdot\hat{\mathscr{r}}/c\right)\right)^2}$$
$$=\frac{q}{4\pi\epsilon_0}\frac{\frac{1}{c}\vec{v}+\frac{1}{c}\left(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a}\right)\nabla t_r}{\left(\mathscr{r}\left(1-\vec{v}\cdot\hat{\mathscr{r}}/c\right)\right)^2}$$

Now, we shell find the gradient ∇t_r.

Since

$$r^2 = \left(\vec{r} - \vec{w}\right)^2$$
$$2r\nabla r = 2\left(\vec{r} - \vec{w}\right) \cdot \left(\nabla\vec{r} - \nabla\vec{w}\right) = 2\left(\vec{r} - \vec{w}\right) \cdot \left(1 - \vec{v}\nabla t_r\right)$$
$$\Rightarrow r\left(-c\nabla t_r\right) = \vec{r} \cdot \left(1 - \vec{v}\nabla t_r\right)$$
$$\Rightarrow \left(\vec{r} \cdot \vec{v}\right)\nabla t_r - rc\nabla t_r = \vec{r}$$
$$\Rightarrow \nabla t_r = \frac{\vec{r}}{\left(\vec{r} \cdot \vec{v}\right) - rc}$$

we obtain

$$\nabla r = -c\frac{\vec{r}}{\left(\vec{r} \cdot \vec{v}\right) - rc}$$
$$\nabla\left(\vec{v} \cdot \vec{r}\right) = \vec{v} + \left(\left(\vec{r} \cdot \vec{a}\right) - v^2\right)\frac{\vec{r}}{\left(\vec{r} \cdot \vec{v}\right) - rc}$$

Substituting into the electric field, we obtain

$$\nabla\varphi = \frac{q}{4\pi\epsilon_0}\frac{\frac{1}{c}\vec{v} + \frac{1}{c}\left(c^2 - v^2 + \vec{r} \cdot \vec{a}\right)\frac{\vec{r}}{\left(\vec{r} \cdot \vec{v}\right) - rc}}{\frac{1}{c^2}\left(rc - \vec{v} \cdot \vec{r}\right)^2}$$
$$= \frac{qc}{4\pi\epsilon_0}\frac{\vec{v} - \left(c^2 - v^2 + \vec{r} \cdot \vec{a}\right)\frac{\vec{r}}{\left(rc - \vec{v} \cdot \vec{r}\right)}}{\left(rc - \vec{v} \cdot \vec{r}\right)^2}$$
$$= \frac{qc}{4\pi\epsilon_0}\frac{\left(rc - \vec{v} \cdot \vec{r}\right)\vec{v} - \left(c^2 - v^2 + \vec{r} \cdot \vec{a}\right)\vec{r}}{\left(rc - \vec{v} \cdot \vec{r}\right)^2}$$

(3) The differentiate $\vec{A}\left(\vec{r}, t\right)$ with respect to t

$$
\begin{aligned}
\frac{\partial\vec{A}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\vec{v}(t_r)}{c^2}\varphi(\vec{r},t)\right)\\
&= \frac{1}{c^2}\left(\frac{\partial\vec{v}(t_r)}{\partial t}\varphi(\vec{r},t)+\vec{v}(t_r)\frac{\partial\varphi(\vec{r},t)}{\partial t}\right)\\
&= \frac{1}{c^2}\left(\frac{\partial\vec{v}(t_r)}{\partial t_r}\frac{\partial t_r}{\partial t}\varphi(\vec{r},t)+\vec{v}(t_r)\frac{\partial\varphi(\vec{r},t)}{\partial t}\right)\\
&= \frac{1}{c^2}\left[\vec{a}\frac{\partial t_r}{\partial t}\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})}+\frac{qc}{4\pi\epsilon_0}\vec{v}\frac{-\frac{\partial}{\partial t}(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})}{(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 c(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^2}\Bigg[\vec{a}(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\frac{\partial t_r}{\partial t}\\
&\qquad -\vec{v}\left(c\frac{\partial\mathscr{r}}{\partial t}-\frac{\partial\vec{v}}{\partial t}\cdot\vec{\mathscr{r}}-\vec{v}\cdot\frac{\partial\vec{\mathscr{r}}}{\partial t}\right)\Bigg]
\end{aligned}
$$

Since

$$\mathscr{r} = c(t-t_r)$$

we obtain

$$
\begin{aligned}
&\frac{\partial\mathscr{r}}{\partial t} = c\left(1-\frac{\partial t_r}{\partial t}\right)\\
&\frac{\partial\vec{v}}{\partial t} = \frac{\partial\vec{v}}{\partial t_r}\frac{\partial t_r}{\partial t} = \vec{a}\frac{\partial t_r}{\partial t}\\
&\frac{\partial\vec{\mathscr{r}}}{\partial t} = -\frac{\partial\vec{w}}{\partial t} = -\frac{\partial\vec{w}}{\partial t_r}\frac{\partial t_r}{\partial t} = -\vec{v}\frac{\partial t_r}{\partial t}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial\vec{A}}{\partial t} &= \frac{q}{4\pi\epsilon_0 c(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^2}\Bigg[\vec{a}(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\frac{\partial t_r}{\partial t}\\
&\qquad -\vec{v}\left(c^2\left(1-\frac{\partial t_r}{\partial t}\right)-\vec{\mathscr{r}}\cdot\vec{a}\frac{\partial t_r}{\partial t}+v^2\frac{\partial t_r}{\partial t}\right)\Bigg]\\
&= \frac{q}{4\pi\epsilon_0 c(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^2}\Bigg[-c^2\vec{v}\\
&\qquad +\left((\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\vec{a}+(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right)\frac{\partial t_r}{\partial t}\Bigg]
\end{aligned}
$$

Here

$$\frac{\partial t_r}{\partial t} = 1-\frac{\partial}{\partial t}\frac{\mathscr{r}}{c} = 1-\frac{1}{c}\frac{\partial\mathscr{r}}{\partial t}$$

Since

$$\vec{\mathscr{r}} \cdot \vec{\mathscr{r}} = c^2 (t - t_r)^2$$

$$\Rightarrow 2\vec{\mathscr{r}} \cdot \frac{\partial \vec{\mathscr{r}}}{\partial t} = 2c^2 (t - t_r) \left(1 - \frac{\partial t_r}{\partial t}\right)$$

Since

$$\frac{\partial \vec{\mathscr{r}}}{\partial t} = -\frac{\partial \vec{w}}{\partial t} = -\frac{\partial \vec{w}}{\partial t_r} \frac{\partial t_r}{\partial t} = -\vec{v} \frac{\partial t_r}{\partial t}$$

$$\Rightarrow -\vec{\mathscr{r}} \cdot \vec{v} \frac{\partial t_r}{\partial t} = \mathscr{r} c \left(1 - \frac{\partial t_r}{\partial t}\right)$$

$$\Rightarrow \frac{\partial t_r}{\partial t} = \frac{\mathscr{r} c}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}}$$

Thus, we obtain

$$\frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_0 c (\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}})^2} \left[- c^2 \vec{v} + \left((\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}}) \vec{a} + (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a}) \vec{v} \right) \frac{\mathscr{r} c}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}} \right]$$

$$= \frac{qc}{4\pi\epsilon_0 (\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}})^3} \left[- \vec{v} (\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}}) + \left((\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}}) \vec{a} + (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a}) \vec{v} \right) \frac{\mathscr{r}}{c} \right]$$

$$= \frac{qc}{4\pi\epsilon_0 (\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}})^3} \left[(\mathscr{r} c - \vec{v} \cdot \vec{\mathscr{r}}) \left(-\vec{v} + \frac{\mathscr{r}}{c} \vec{a} \right) + \frac{\mathscr{r}}{c} (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a}) \vec{v} \right]$$

(4) We then obtain the electric field as

$$
\begin{aligned}
\vec{E} &= -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{qc}{4\pi\epsilon_0}\frac{(rc-\vec{v}\cdot\vec{r})\vec{v}-(c^2-v^2+\vec{r}\cdot\vec{a})\vec{r}}{(rc-\vec{v}\cdot\vec{r})^3} \\
&\quad -\frac{qc}{4\pi\epsilon_0(rc-\vec{v}\cdot\vec{r})^3}\left[(rc-\vec{v}\cdot\vec{r})\left(-\vec{v}+\frac{r}{c}\vec{a}\right)\right. \\
&\qquad \left. +\frac{r}{c}(c^2-v^2+\vec{r}\cdot\vec{a})\vec{v}\right] \\
&= \frac{qc}{4\pi\epsilon_0(rc-\vec{v}\cdot\vec{r})^3}\left[(c^2-v^2)\left(\vec{r}-\frac{r}{c}\vec{v}\right)\right. \\
&\qquad \left. +(\vec{r}\cdot\vec{a})\left(\vec{r}-\frac{r}{c}\vec{v}\right)-(rc-\vec{v}\cdot\vec{r})\frac{r}{c}\vec{a}\right] \\
&= \frac{qc}{4\pi\epsilon_0(rc-\vec{v}\cdot\vec{r})^3}\left[(c^2-v^2)\left(\vec{r}-\frac{r}{c}\vec{v}\right)\right. \\
&\qquad \left. +\underbrace{\left(\vec{r}-\frac{r}{c}\vec{v}\right)(\vec{a}\cdot\vec{r})-\vec{a}\left(\vec{r}\cdot\left(\vec{r}-\frac{r}{c}\vec{v}\right)\right)}_{\text{BAC-CAB rule}}\right] \\
&= \frac{qc}{4\pi\epsilon_0(rc-\vec{v}\cdot\vec{r})^3}\left[(c^2-v^2)\left(\vec{r}-\frac{r}{c}\vec{v}\right)+\vec{r}\times\left(\vec{r}-\frac{r}{c}\vec{v}\right)\times\vec{a}\right]
\end{aligned}
$$

(5) The curl of $\vec{A}(\vec{r},t)$ is

$$
\begin{aligned}
\nabla\times\vec{A} &= \nabla\times\frac{\vec{v}(t_r)}{c^2}\varphi(\vec{r},t) \\
&= \frac{1}{c^2}\left(\varphi\nabla\times\vec{v}-\vec{v}\times\nabla\varphi\right) \\
&= \frac{1}{c^2}\left[\frac{q}{4\pi\epsilon_0 \mathscr{r}\left(1-\vec{v}\cdot\hat{\mathscr{r}}/c\right)}\left(-\vec{a}\times\frac{\vec{\mathscr{r}}}{(\vec{\mathscr{r}}\cdot\vec{v})-\mathscr{r}c}\right)\right. \\
&\qquad\left. -\vec{v}\times\frac{qc}{4\pi\epsilon_0}\frac{(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\vec{v}-(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}}}{(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}\right] \\
&= \frac{1}{c}\frac{q}{4\pi\epsilon_0(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}[(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})(\vec{a}\times\vec{\mathscr{r}}) \\
&\qquad +(c^2-v^2+\vec{\mathscr{r}}\cdot\vec{a})(\vec{v}\times\vec{\mathscr{r}})] \\
&= -\frac{1}{c}\frac{q}{4\pi\epsilon_0(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}\vec{\mathscr{r}}\times[(c^2-v^2)\vec{v} \\
&\qquad +\vec{v}(\vec{\mathscr{r}}\cdot\vec{a})+(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\vec{a}]
\end{aligned}
$$

Since

$$
\begin{aligned}
\hat{\mathscr{r}}\times\vec{E} &= \frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}\hat{\mathscr{r}}\times\left[(c^2-v^2)\left(\vec{\mathscr{r}}-\frac{\mathscr{r}}{c}\vec{v}\right)\right. \\
&\qquad\left. +\left(\vec{\mathscr{r}}-\frac{\mathscr{r}}{c}\vec{v}\right)(\vec{\mathscr{r}}\cdot\vec{a})-\vec{a}\left(\vec{\mathscr{r}}\cdot\left(\vec{\mathscr{r}}-\frac{\mathscr{r}}{c}\vec{v}\right)\right)\right] \\
&= -\frac{qc}{4\pi\epsilon_0(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}\hat{\mathscr{r}}\times\left[(c^2-v^2)\frac{\mathscr{r}}{c}\vec{v}\right. \\
&\qquad\left. +\frac{\mathscr{r}}{c}\vec{v}(\vec{\mathscr{r}}\cdot\vec{a})+\vec{a}\left(\vec{\mathscr{r}}\cdot\left(\vec{\mathscr{r}}-\frac{\mathscr{r}}{c}\vec{v}\right)\right)\right] \\
&= -\frac{q}{4\pi\epsilon_0(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})^3}\vec{\mathscr{r}}\times\left[(c^2-v^2)\vec{v}\right. \\
&\qquad\left. +\vec{v}(\vec{\mathscr{r}}\cdot\vec{a})+(\mathscr{r}c-\vec{v}\cdot\vec{\mathscr{r}})\vec{a}\right] \\
&= c\nabla\times\vec{A}
\end{aligned}
$$

Thus, we obtain the magnetic field as

$$
\vec{B}(\vec{r},t)=\nabla\times\vec{A}(\vec{r},t)=\frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)
$$

⇒ The magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.

EXAMPLES:

1. Find the electric and magnetic fields of a point charge moving with constant velocity.
ANSWER:

$$\vec{a} = 0$$

$$\vec{E} = \frac{qc}{4\pi\epsilon_0\left(\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}}\right)^3}\left[\left(c^2 - v^2\right)\left(\vec{\mathscr{r}} - \frac{\mathscr{r}}{c}\vec{v}\right)\right]$$

Using $\vec{w} = \vec{v}t$, we obtian

$$\left(\vec{\mathscr{r}} - \frac{\mathscr{r}}{c}\vec{v}\right) = \left[\left(\vec{r} - \vec{v}t_r\right) - \frac{c\left(t - t_r\right)}{c}\vec{v}\right]$$

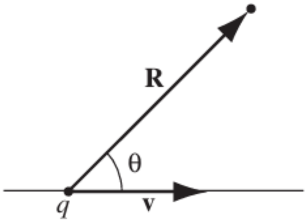
$$= \vec{r} - \vec{v}t$$

$$\left(\mathscr{r}c - \vec{v}\cdot\hat{\mathscr{r}}\right) = c^2\left(t - t_r\right) - \vec{v}\cdot\left(\vec{r} - \vec{v}t_r\right)$$

$$= \left(c^2t - \vec{v}\cdot\vec{r}\right) - \left(c^2 - v^2\right)t_r$$

$$= \sqrt{\left(c^2t - \vec{r}\cdot\vec{v}\right)^2 + (c^2 - v^2)(r^2 - c^2t^2)}$$

Let $\vec{R} = \vec{r} - \vec{v}$ be the vector from the present position of the particle to the field point $\vec{r}$, and θ is the angle between $\vec{R}$ and $\vec{v}$.



We obtain

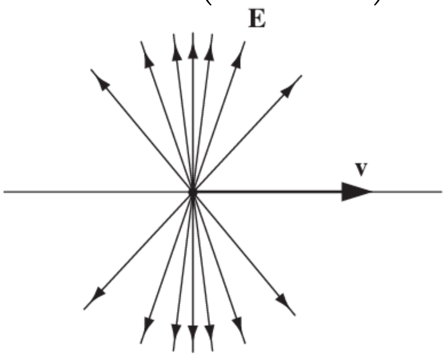
$$\sqrt{\left(c^2t - \vec{r}\cdot\vec{v}\right)^2 + (c^2 - v^2)(r^2 - c^2t^2)} = \sqrt{\mathrm{c}^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)}$$

The electric field is

$$\vec{E} = \frac{qc}{4\pi\epsilon_0 \left(c^2 R^2 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right)\right)^{3/2}} (c^2 - v^2)\vec{R}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{c(c^2 - v^2)}{c^3 R^3 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \vec{R}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2}$$



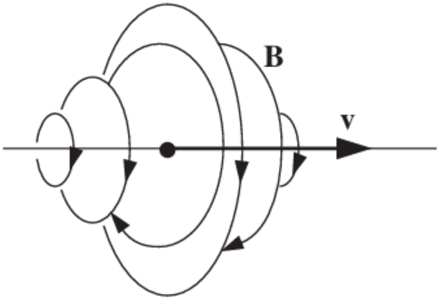
As for $\vec{B}$, we have

$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{v}t_r}{\mathscr{r}} = \frac{(\vec{r} - \vec{v}t) + (t - t_r)\vec{v}}{\mathscr{r}} = \frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}}{c}$$

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}} \times \vec{E}$$

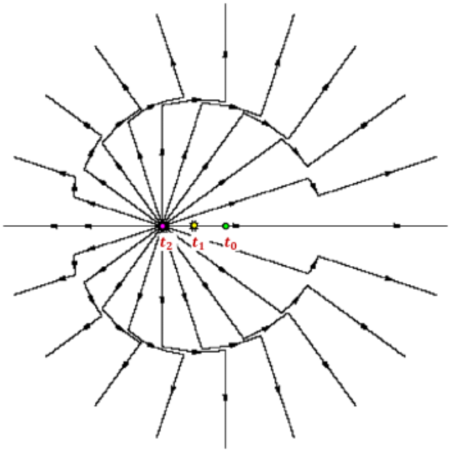
$$= \frac{1}{c}\left(\frac{\vec{R}}{\mathscr{r}} + \frac{\vec{v}}{c}\right) \times \frac{q}{4\pi\epsilon_0} \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2}$$

$$= \frac{1}{c^2}(\vec{v} \times \vec{E})$$



Lines of $\vec{B}$ circle around the charge.

2. Draw the electric fields for an abrupt, momentary acceleration.
ANSWER:



A point electric charge q, initially at rest $t_0 = 0$.
The charge undergoes an abrupt acceleration lasting $\Delta t = t_1 - t_0 = t_1$.
Then, the charge continuous to move to the left with constant velocity v, at time t_2.

13-3 Power Radiated by a Point Charge

A. RADIATION

(1) The fields of a point charge in arbitrary motion

$$\vec{E} = \frac{qc}{4\pi\epsilon_0\left(\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}}\right)^3}\left[\left(c^2 - v^2\right)\left(\hat{\mathscr{r}} - \frac{\mathscr{r}}{c}\vec{v}\right) + \vec{\mathscr{r}} \times \left(\hat{\mathscr{r}} - \frac{\mathscr{r}}{c}\vec{v}\right) \times \vec{a}\right]$$

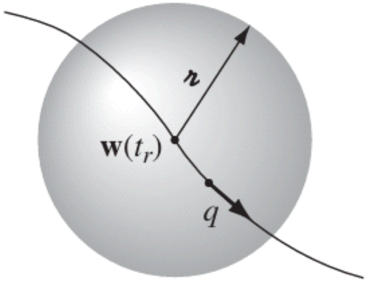
$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}} \times \vec{E}(\vec{r},t)$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \underbrace{\frac{1}{\mu_0 c}\left[\vec{E}\times\left(\hat{\mathscr{r}}\times\vec{E}\right)\right]}_{\text{BAC-CAB rule}} = \frac{1}{\mu_0 c}\left[E^2\hat{\mathscr{r}} - \vec{E}\left(\hat{\mathscr{r}}\cdot\vec{E}\right)\right]$$

(2) Imagine a huge sphere of radius $\mathscr{r}$, centered at the position of the charge at retarded time t_r, i.e., the time interval for the fields to reach the sphere,

$$t - t_r = \frac{\mathscr{r}}{c}$$



The total power passing through the surface is the integral of the Poynting vector:

$$P(\mathscr{r},t) = \oint \vec{S}\cdot d\vec{a} = \frac{1}{\mu_0}\oint\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a}$$

Thus, the power is carried away out to infinity and never comes back,

$$P_{\text{rad}}(t_r) = \lim_{\mathscr{r}\to\infty} P\left(\mathscr{r}, t_r + \frac{\mathscr{r}}{c}\right)$$

such a process called *radiation*.

(3) Since the area of the sphere is $4\pi\mathscr{r}^2$, so any term in $\vec{S}$ that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r}\to\infty$. For this reason, $\vec{E}$ and $\vec{B}$ go like

$1/r$ at large distance from the source, constructing from them the $1/r^2$ term in $\vec{S}$. Thus, we find that

$$\vec{E} = \frac{qc}{4\pi\epsilon_0 \underbrace{\left(rc - \vec{v}\cdot\vec{r}\right)^3}_{\sim 1/r^3}} \left[\underbrace{\left(c^2 - v^2\right)\left(\vec{r} - \frac{r}{c}\vec{v}\right)}_{\sim r} + \underbrace{\vec{r} \times \left(\vec{r} - \frac{r}{c}\vec{v}\right) \times \vec{a}}_{\sim r^2}\right]$$

the second term (depending on the acceleration) falls of as $1/r$ and is therefore dominant at large distance.

$$\vec{E}_{\text{rad}} = \frac{qc}{4\pi\epsilon_0\left(rc - \vec{v}\cdot\vec{r}\right)^3}\left[\vec{r} \times \left(\vec{r} - \frac{r}{c}\vec{v}\right) \times \vec{a}\right]$$
$$= \frac{q}{4\pi\epsilon_0 c^2 r\left(1 - \frac{\vec{v}\cdot\hat{r}}{c}\right)^3}\left[\hat{r} \times \left(\hat{r} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]$$

Now, $\vec{E}_{\text{rad}}$ is perpendicular to $\hat{r}$, so the second term in $\vec{S}$ vanishes, i.e.,

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left[E^2\hat{r} - \underbrace{\vec{E}\left(\hat{r}\cdot\vec{E}\right)}_{=0}\right]$$
$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2 r}\right)^2 \frac{\left[\hat{r} \times \left(\hat{r} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]^2}{\left(1 - \frac{\vec{v}\cdot\hat{r}}{c}\right)^6}\hat{r}$$
$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\hat{r} \times \left(\hat{r} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]^2}{\left(1 - \frac{\vec{v}\cdot\hat{r}}{c}\right)^6}\frac{\hat{r}}{r^2}$$

⇒ When charges **accelerate**, their fields are radiated.

(4) If the charge is instantaneously at rest $(\vec{v} = 0)$ or $v \ll c$, we obtain the Poynting vector

$$
\begin{aligned}
\vec{S}_{\text{rad}} &= \frac{\mu_0 q^2}{16\pi^2 c}\left[\hat{r}\times\hat{r}\times\vec{a}\right]^2\frac{\hat{r}}{r^2} \\
&= \frac{\mu_0 q^2}{16\pi^2 c}\left[\hat{r}(\hat{r}\cdot\vec{a})-\vec{a}\right]^2\frac{\hat{r}}{r^2}, \qquad \text{(BAC-CAB rule)} \\
&= \frac{\mu_0 q^2}{16\pi^2 c}\left[(\hat{r}\cdot\vec{a})^2+a^2-2(\hat{r}\cdot\vec{a})^2\right]\frac{\hat{r}}{r^2} \\
&= \frac{\mu_0 q^2}{16\pi^2 c}\left[a^2-(\hat{r}\cdot\vec{a})^2\right]\frac{\hat{r}}{r^2}
\end{aligned}
$$

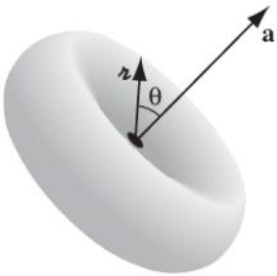
Since

$$
a^2-\underbrace{(\hat{r}\cdot\vec{a})^2}_{a^2\cos^2\theta} = a^2(1-\cos^2\theta) = a^2\sin^2\theta
$$

we obtain

$$
\vec{S}_{\text{rad}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}
$$

The power is radiated in a donut about the direction of instantaneous acceleration.



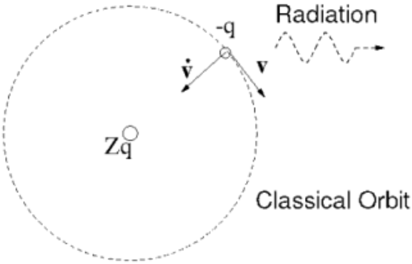
The total power radiated is

$$
\begin{aligned}
P &= \oint \vec{S}_{\text{rad}}\cdot d\vec{a} \\
&= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta d\phi \\
&= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{=4/3} \cdot 2\pi \\
&= \frac{\mu_0 q^2 a^2}{6\pi c} \cdots\cdots \text{ Larmor formula}
\end{aligned}
$$

EXAMPLES:

1. In the Rutherfold model of hydrogen atom, the electron is circulating around the nucleus, and continuously emit energy and spirally fail on the nucleus. Find the energy loss for an electron in

a circular motion per revolution.



ANSWER:

For a circular motion, we have

$$a = \frac{v^2}{r}$$

Using Larmor formula, we obtain

$$P = \frac{\mu_0 e^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 e^2 v^4}{6\pi c r^2}$$

Energy loss for an electron in a circular motion per revolution is

$$\Delta E = \frac{2\pi r}{v} P = \frac{e^2 v^3}{3\epsilon_0 r}$$

B. ANGULAR POWER DISTRIBUTION

(1) The angular distribution of the radiated power at observer's current time t is given by

$$dP(t) = \vec{S}_{\text{rad}} \cdot d\vec{a} = \vec{S}_{\text{rad}} \cdot \mathscr{r}^2 d\Omega \hat{\mathscr{r}}$$

$$\Rightarrow \frac{dP(t)}{d\Omega} = \left(\vec{S}_{\text{rad}} \cdot \hat{\mathscr{r}}\right) \mathscr{r}^2$$

(2) The angular distribution of the radiated power as measured with respect to the charge's retarded time t' is given by

$$\frac{dP(t_r)}{d\Omega} = \frac{dP(t)}{d\Omega}\frac{dt}{dt_r} = \left(\vec{S}_{\text{rad}} \cdot \hat{\mathscr{r}}\right) \mathscr{r}^2 \frac{dt}{dt_r}$$

Since

$$\frac{\partial t_r}{\partial t} = \frac{\mathscr{r} c}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}}$$

we obtain

$$\frac{dP(t_r)}{d\Omega} = \left(\vec{S}_{\text{rad}} \cdot \hat{\mathscr{r}}\right) \mathscr{r}^2 \frac{\left(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v}\right)}{\mathscr{r}c}$$
$$= \left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right) \left(\vec{S}_{\text{rad}} \cdot \hat{\mathscr{r}}\right) \mathscr{r}^2$$
$$= \left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right) \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\hat{\mathscr{r}} \times \left(\hat{\mathscr{r}} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]^2}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^6} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \hat{\mathscr{r}} \mathscr{r}^2$$
$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\hat{\mathscr{r}} \times \left(\hat{\mathscr{r}} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]^2}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^5}$$

EXAMPLES:

1. Suppose $\vec{v}$ and $\vec{a}$ are instantaneously collinear (at time t_r), as, for example, in straight-line motion. Find the angular distribution of the radiation and the total power emitted.

   ANSWER:

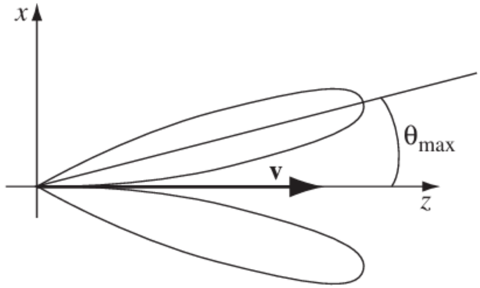
   In this case, $\vec{v} \times \vec{a} = 0$, so

$$\frac{dP(t_r)}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\hat{\mathscr{r}} \times (\hat{\mathscr{r}}) \times \vec{a}\right]^2}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^5}$$
$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\hat{\mathscr{r}}\left(\hat{\mathscr{r}} \cdot \vec{a}\right) - \vec{a}\right]^2}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^5}, \qquad \text{(BAC-CAB rule)}$$
$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[\left(\hat{\mathscr{r}} \cdot \vec{a}\right)^2 + a^2 - 2\left(\hat{\mathscr{r}} \cdot \vec{a}\right)^2\right]}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^5}$$
$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{\left[a^2 - \left(\hat{\mathscr{r}} \cdot \vec{a}\right)^2\right]}{\left(1 - \frac{\vec{v} \cdot \hat{\mathscr{r}}}{c}\right)^5}$$

   If we let the z-axis point along $\vec{v}$, then

$$\frac{dP(t_r)}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{[1-\cos^2\theta]}{\left(1-\frac{v}{c}\cos\theta\right)^5} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5}$$

For very large v, the distribution becomes



Although there is still no radiation in precisely the forward direction, most of it is concentrated within an increasingly narrow cone about the forward direction.

The total power emitted is

$$P = \int \frac{dP(t_r)}{d\Omega} d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} \sin\theta \, d\theta d\phi$$

Since

$$\begin{aligned}\int_0^\pi \frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} \sin\theta \, d\theta &= \int_{-1}^{1} \frac{1-\cos^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} d(\cos\theta) \\ &= \frac{4}{3}\frac{1}{(1-v^2/c^2)^3} \\ &= \frac{4}{3}\gamma^6\end{aligned}$$

we obtain

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \cdot \frac{4}{3}\gamma^6 \cdot 2\pi = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}$$

2. For v close to c, find the angle $\theta_{\max}$ and the intensity of the radiation in this maximal direction.

ANSWER:

The maximum angle occurs at

$$\frac{d}{d\theta} \frac{\sin^2\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} = 0$$

$$\Rightarrow \frac{2\sin\theta\cos\theta}{\left(1-\frac{v}{c}\cos\theta\right)^5} - \frac{\sin^2\theta\, 5\frac{v}{c}\sin\theta}{\left(1-\frac{v}{c}\cos\theta\right)^6} = 0$$

$$\Rightarrow \cos\theta = \frac{-2 \pm \sqrt{4+60\,v^2/c^2}}{6\,v/c} = \frac{-1 \pm \sqrt{1+15\,v^2/c^2}}{3\,v/c}$$

Since $\theta_{\max} \to 90^\circ$ when $v \to 0$, we pick up the plus sign.

$$\theta_{\max} = \cos^{-1}\left(\frac{\sqrt{1+15\,v^2/c^2}-1}{3\,v/c}\right)$$

For $v \approx c$, let $v/c = 1-\epsilon$, and expand to the first order in ϵ:

$$\begin{aligned}\frac{\sqrt{1+15\,v^2/c^2}-1}{3\,v/c} &= \frac{1}{3(1-\epsilon)}\left[\sqrt{1+15(1-\epsilon)^2}-1\right] \\ &\approx \frac{1}{3}(1+\epsilon)\left[\sqrt{1+15(1-2\epsilon)}-1\right] \\ &\approx \frac{1}{3}(1+\epsilon)\left[4\left(1-\frac{1}{2}\cdot\frac{15}{8}\epsilon\right)-1\right] \\ &= (1+\epsilon)\left(1-\frac{5}{4}\epsilon\right) \\ &\approx 1+\epsilon-\frac{5}{4}\epsilon \\ &= 1-\frac{1}{4}\epsilon\end{aligned}$$

$$\Rightarrow \theta_{\max} = \cos^{-1}\left(1-\frac{1}{4}\epsilon\right) \approx 0$$

Thus, we obtain

$$\cos\theta_{\max} \approx 1-\frac{1}{2}\theta_{\max}^2 = 1-\frac{1}{4}\epsilon$$

$$\Rightarrow \theta_{\max} = \sqrt{\frac{\epsilon}{2}} = \sqrt{\frac{1-v/c}{2}}$$

The intensity of the radiation in this maximal direction

$$\frac{dP(t_r)}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta_{\max}}{\left(1-\frac{v}{c}\cos\theta_{\max}\right)^5}$$

Since

$$\sin^2\theta_{\max} \approx \frac{\epsilon}{2}$$

$$1-\frac{v}{c}\cos\theta_{\mathrm{max}} \approx 1-(1-\epsilon)\left(1-\frac{1}{4}\epsilon\right) \approx \epsilon+\frac{1}{4}\epsilon=\frac{5}{4}\epsilon$$

we obtain

$$\frac{dP(t_r)}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\epsilon/2}{(5\epsilon/4)^5}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{4}{5}\right)^5\frac{1}{2\epsilon^4}$$

Since

$$\gamma^2=\frac{1}{1-v^2/c^2} \approx \frac{1}{1-(1-\epsilon)^2} \approx \frac{1}{2\epsilon}$$

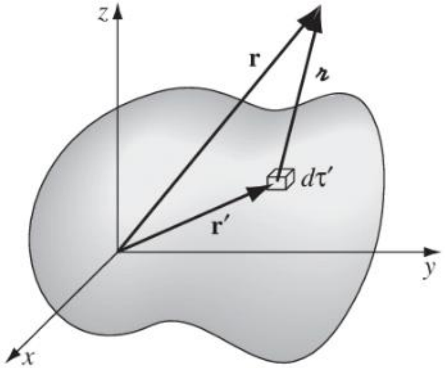
we obtain

$$\frac{dP(t_r)}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{4}{5}\right)^5\frac{1}{2}\left(2\gamma^2\right)^4=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\times 2.62\gamma^8$$

13-4 Radiation from an Arbitrary Source

A. RADIATION FROM AN ARBITRARY SOURCE

(1) The retarded scalar potential of an arbitrary configuration of charge is



$$\varphi(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}d\tau'$$

where

$$t_r=t-\frac{\mathscr{r}}{c}$$

$$\mathscr{r}=|\vec{r}-\vec{r}\,'|=\sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}\,'}\approx r-\hat{r}\cdot\vec{r}\,'$$

Write $1/\mathscr{r}$ in the form of a power series with Legendre polynomials:

$$\frac{1}{\mathscr{r}}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)=\frac{1}{r}+\frac{\hat{r}\cdot\vec{r}\,'}{r^2}+\cdots$$

We obtain

$$\rho(\vec{r}\,',t_r)=\rho\left(\vec{r}\,',t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)$$

Let

$$t_0=t-\frac{r}{c}$$

We have

$$\rho\left(\vec{r}\,',t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)$$

$$\approx\rho(\vec{r}\,',t_0)+\dot{\rho}(\vec{r}\,',t_0)\left(\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)+\frac{1}{2}\ddot{\rho}(\vec{r}\,',t_0)\left(\frac{\hat{r}\cdot\vec{r}\,'}{c}\right)^2$$

The retarded scalar potential becomes

$$\varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left[\int \frac{\rho(\vec{r}',t_0)}{r} d\tau' + \int \frac{\rho(\vec{r}',t_0)}{r^2}(\hat{r}\cdot\vec{r}')\, d\tau' + \int \frac{\dot{\rho}(\vec{r}',t_0)}{r}\left(\frac{\hat{r}\cdot\vec{r}'}{c}\right) d\tau' + \cdots\right]$$

$$\approx \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right]$$

where $\vec{p}$ is the dipole moment.

(2) The retarded vector potential of an arbitrary configuration of current is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{r} d\tau'$$

$$\approx \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)}{r} d\tau'$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\int \vec{J}(\vec{r}',t_0)\, d\tau'$$

Since for a configuration of charges and currents confined within a volume $\mathcal{V}$, the integral of $\vec{J}$ is the time derivative of the dipole moment.

$$\int_{\mathcal{V}} \vec{J}\, d\tau = \frac{d\vec{p}}{dt}$$

the vector potential becomes

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\frac{d\vec{p}(t_0)}{dt} = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_0)}{r}$$

PROOF:

The time derivative of the dipole moment,

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\int_{\mathcal{V}} \rho\vec{r}\, d\tau = \int_{\mathcal{V}} \frac{\partial\rho}{\partial t}\vec{r}\, d\tau$$

Using the continuity equation:

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

We obtain

$$\frac{d\vec{p}}{dt} = -\int_{\mathcal{V}} \left(\nabla\cdot\vec{J}\right)\vec{r}\, d\tau$$

Since

$$\nabla\cdot\left(r\vec{J}\right) = r\left(\nabla\cdot\vec{J}\right) + \vec{J}\cdot(\nabla r) = r\left(\nabla\cdot\vec{J}\right) + \vec{J}\cdot\hat{r}$$

we obtain

$$\int r\left(\nabla\cdot\vec{J}\right)d\tau=\int_{\mathcal{V}}\nabla\cdot\left(r\vec{J}\right)d\tau-\int\vec{J}\cdot\hat{r}\,d\tau=\int_{\mathcal{S}}r\vec{J}\cdot d\vec{a}-\int J_r\,d\tau$$

Since $\vec{J}$ is entirely inside $\mathcal{V}$, it is zero on the surface $\mathcal{S}$. Therefore, we have

$$\int r\left(\nabla\cdot\vec{J}\right)d\tau=-\int J_r\,d\tau\Rightarrow\int_{\mathcal{V}}\left(\nabla\cdot\vec{J}\right)\vec{r}\,d\tau=-\int\vec{J}\,d\tau$$

Thus, we obtain

$$\int_{\mathcal{V}}\vec{J}\,d\tau=\frac{d\vec{p}}{dt}$$

■

(3) Therefore, the fields are

$$\begin{aligned}\vec{E}&=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}\\&=-\frac{1}{4\pi\epsilon_0}\nabla\left[\frac{Q}{r}+\frac{\hat{r}\cdot\vec{p}(t_0)}{r^2}+\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right]-\frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}\\&=\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r^2}+\frac{\hat{r}\cdot\vec{p}(t_0)}{r^3}-\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{r^2}\nabla t_0+\frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{r^2c}-\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{rc}\nabla t_0\right]\\&\qquad-\frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}\end{aligned}$$

$$\vec{B}=\nabla\times\vec{A}=\frac{\mu_0}{4\pi r}\nabla\times\dot{\vec{p}}(t_0)=\frac{\mu_0}{4\pi r}\nabla t_0\times\ddot{\vec{p}}(t_0)$$

Only the terms involving the acceleration have the contribution. Thus, we obtain

$$\vec{E}_{\text{rad}}=-\frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{rc}\nabla t_0-\frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r}$$

$$\vec{B}_{\text{rad}}=\frac{\mu_0}{4\pi r}\nabla t_0\times\ddot{\vec{p}}(t_0)$$

Since

$$\nabla t_0=\nabla\left(t-\frac{r}{c}\right)=-\frac{\hat{r}}{c}$$

the radiation fields are

$$\begin{aligned}\vec{E}_{\text{rad}} &= \frac{1}{4\pi\epsilon_0 c^2}\frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{r}\hat{r} - \frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}(t_0)}{r} \\ &= \frac{\mu_0}{4\pi}\frac{1}{r}\left[\hat{r}\cdot\ddot{\vec{p}}(t_0)\hat{r} - \ddot{\vec{p}}(t_0)\right] \\ &= \frac{\mu_0}{4\pi}\frac{\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}\right)\right]}{r}, \qquad \text{(BAC-CAB rule)} \\ \vec{B}_{\text{rad}} &= -\frac{\mu_0}{4\pi c}\frac{\hat{r}\times\ddot{\vec{p}}(t_0)}{r} = \frac{1}{c}\hat{r}\times\vec{E}\end{aligned}$$

(4) The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = \frac{1}{\mu_0 c}\underbrace{\left[\vec{E}\times\left(\hat{r}\times\vec{E}\right)\right]}_{\text{BAC-CAB rule}} = \frac{1}{\mu_0 c}\left[E^2\hat{r} - \vec{E}\left(\hat{r}\cdot\vec{E}\right)\right]$$

Since $\vec{E}_{\text{rad}}$ is perpendicular to $\hat{r}$, so the second term in $\vec{S}_{\text{rad}}$ vanishes. Thus, we obtain

$$\begin{aligned}\vec{S}_{\text{rad}} &= \frac{1}{\mu_0 c}E_{\text{rad}}^2\hat{r} \\ &= \frac{1}{\mu_0 c}\left(\frac{\mu_0}{4\pi}\right)^2\frac{\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}\right)\right]^2}{r^2}\hat{r} \\ &= \frac{\mu_0}{16\pi^2 c}\left[\hat{r}\left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right) - \ddot{\vec{p}}(t_0)\right]^2\frac{\hat{r}}{r^2}, \qquad \text{(BAC-CAB rule)} \\ &= \frac{\mu_0}{16\pi^2 c}\left[\left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)^2 - 2\left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)^2 + \left(\ddot{p}(t_0)\right)^2\right]\frac{\hat{r}}{r^2} \\ &= \frac{\mu_0}{16\pi^2 c}\left[\left(\ddot{p}(t_0)\right)^2 - \left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)^2\right]\frac{\hat{r}}{r^2}\end{aligned}$$

The radiated power is

$$P_{\text{rad}}(t_0) = \oint \vec{S}_{\text{rad}}\cdot d\vec{a} = \frac{\mu_0}{16\pi^2 c}\oint\left[\left(\ddot{p}(t_0)\right)^2 - \left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)^2\right]\frac{\hat{r}}{r^2}\cdot d\vec{a}$$

(5) If we use spherical coordinates, with the z-axis in the direction of $\ddot{\vec{p}}(t_0)$, we have

$$\hat{r}\cdot\ddot{\vec{p}}(t_0) = \ddot{p}(t_0)\cos\theta$$

The radiation fields are

$$\vec{E}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}\right)\right]}{r}$$

$$\vec{B}_{\text{rad}} = -\frac{\mu_0}{4\pi c}\frac{\hat{r}\times\ddot{\vec{p}}(t_0)}{r} = \frac{1}{c}\hat{r}\times\vec{E}$$

The Poynting vector is

$$\vec{S}_{\text{rad}} = \frac{\mu_0}{16\pi^2 c}\left[\left(\ddot{p}(t_0)\right)^2 - \left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)^2\right]\frac{\hat{r}}{r^2}$$

$$= \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2\left[1-\cos^2\theta\right]\frac{\hat{r}}{r^2}$$

$$= \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2\sin^2\theta\,\frac{\hat{r}}{r^2}$$

The total power radiated is

$$P_{\text{rad}}(t_0) = \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta d\phi$$

$$= \frac{\mu_0}{6\pi c}\left(\ddot{p}(t_0)\right)^2$$

B. DIPOLE RADIATION

(1) The oscillating electric dipole

$$\vec{p}(t) = p_0\cos(\omega t)\,\hat{z}, \qquad (p_0 = qd)$$

$$\dot{\vec{p}}(t) = -\omega p_0\sin(\omega t)\,\hat{z}$$

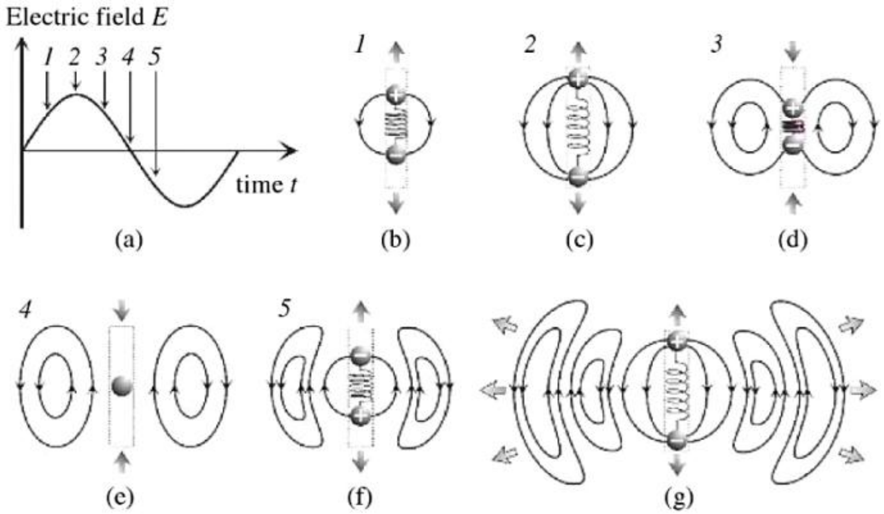
$$\ddot{\vec{p}}(t) = -\omega^2 p_0\cos(\omega t)\,\hat{z}$$

The electric and magnetic fields are

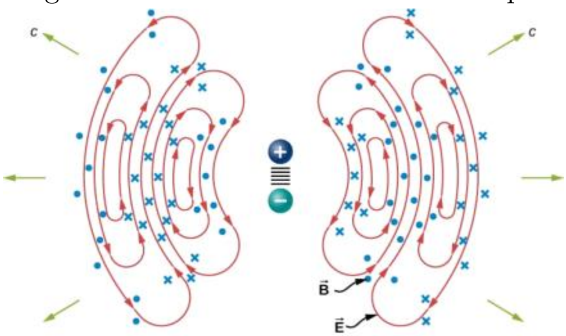
$$\vec{E}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{\left[\hat{r}\times\left(\hat{r}\times\ddot{\vec{p}}\right)\right]}{r} = -\frac{\mu_0 p_0\omega^2}{4\pi}\cos(\omega t_0)\left(\frac{\sin\theta}{r}\right)\hat{\theta}$$

$$\vec{B}_{\text{rad}} = \frac{1}{c}\hat{r}\times\vec{E} = -\frac{\mu_0 p_0\omega^2}{4\pi c}\cos(\omega t_0)\left(\frac{\sin\theta}{r}\right)\hat{\phi}$$

Electric field lines from the electric dipole:



Magnetic field lines from the electric dipole:



The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\vec{S}_{\text{rad}} = \frac{\mu_0}{16\pi^2 c}\left(\ddot{p}(t_0)\right)^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

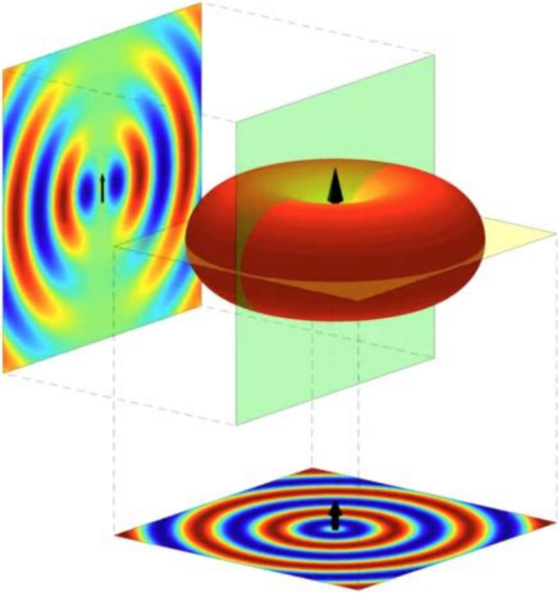
$$= \frac{\mu_0}{16\pi^2 c}\left(-\omega^2 p_0\cos(\omega t_0)\right)^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

$$= \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2(\omega t_0)\,\hat{r}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\left\langle\vec{S}_{\text{rad}}\right\rangle = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\underbrace{\int_0^T \cos^2\omega t_0\,dt_0}_{=1/2}\hat{r} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

The intensity profile takes the form of a donut.



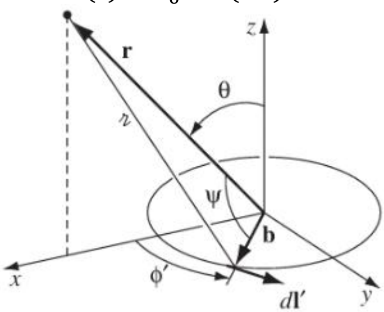
The total power radiated is

$$P_{\text{rad}}(t_0) = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta d\phi$$
$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \cdot \frac{4}{3} \cdot 2\pi$$
$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

(2) The oscillating magnetic dipole

Suppose that we have a wire loop of radius b, around which we derive an alternating current:

$$I(t) = I_0 \cos(\omega t)$$



The loop is uncharged, so the scalar potential is zero. The retarded vector potential is

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\oint\frac{I(t_0)}{\mathscr{r}}d\vec{l}'$$

Since

$$\mathscr{r}=\sqrt{r^2+b^2-2rb\cos\psi}$$
$$\vec{r}=r\sin\theta\,\hat{x}+r\cos\theta\,\hat{z}$$
$$\vec{b}=b\cos\phi'\,\hat{x}+b\sin\phi'\,\hat{y}$$
$$\Rightarrow rb\cos\psi=\vec{r}\cdot\vec{b}=rb\sin\theta\cos\phi'$$

we obtain

$$\mathscr{r}=\sqrt{r^2+b^2-2rb\sin\theta\cos\phi'}\approx r\left(1-\frac{b}{r}\sin\theta\cos\phi'\right)$$
$$\frac{1}{\mathscr{r}}\approx\frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)$$
$$\begin{aligned}I(t_0)&=I\left(t-\frac{\mathscr{r}}{c}\right)\\&\approx I\left(t-\frac{r}{c}+\frac{b}{c}\sin\theta\cos\phi'\right)\\&=I\left(t_0+\frac{b}{c}\sin\theta\cos\phi'\right)\\&\approx I(t_0)+\dot{I}(t_0)\frac{b}{c}\sin\theta\cos\phi'\end{aligned}$$
$$d\vec{l}'=bd\phi'\hat{\phi}=b(-\sin\phi'\,\hat{x}+\cos\phi'\,\hat{y})d\phi'$$

Then

$$\begin{aligned}\vec{A}(\vec{r},t)&=\frac{\mu_0}{4\pi}\oint\frac{1}{r}\left(1+\frac{b}{r}\sin\theta\cos\phi'\right)\left(I(t_0)+\dot{I}(t_0)\frac{b}{c}\sin\theta\cos\phi'\right)\\&\qquad\cdot b(-\sin\phi'\,\hat{x}+\cos\phi'\,\hat{y})d\phi'\\&\approx\frac{\mu_0 b}{4\pi r}\int_0^{2\pi}\left(I(t_0)+\dot{I}(t_0)\frac{b}{c}\sin\theta\cos\phi'+I(t_0)\frac{b}{r}\sin\theta\cos\phi'\right)\\&\qquad\cdot(-\sin\phi'\,\hat{x}+\cos\phi'\,\hat{y})d\phi'\end{aligned}$$

Since

$$\int_0^{2\pi}\sin\phi'\,d\phi'=\int_0^{2\pi}\cos\phi'\,d\phi'=\int_0^{2\pi}\sin\phi'\cos\phi'\,d\phi'=0$$

we obtain

$$\vec{A}(\vec{r},t) = \frac{\mu_0 b}{4\pi r}\int_0^{2\pi}\left(\dot{I}(t_0)\frac{b}{c}\sin\theta\cos^2\phi' + I(t_0)\frac{b}{r}\sin\theta\cos^2\phi'\right)d\phi'\hat{y}$$
$$= \frac{\mu_0 b}{4\pi r}\left[\dot{I}(t_0)\frac{b}{c}\sin\theta + I(t_0)\frac{b}{r}\sin\theta\right]\pi\hat{y}$$
$$= \frac{\mu_0 b^2}{4}\sin\theta\left[\frac{I(t_0)}{r^2} + \frac{\dot{I}(t_0)}{rc}\right]\hat{y}$$

Only $\vec{E}$ and $\vec{B}$ involving the acceleration have the contribution. Thus, we have

$$\vec{A}(\vec{r},t) = \frac{\mu_0 b^2}{4}\sin\theta\frac{\dot{I}(t_0)}{rc}\hat{y}$$

In general $\hat{y} \to \hat{\phi}$

$$\vec{E}_{\text{rad}} = -\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 b^2}{4}\sin\theta\frac{\ddot{I}(t_0)}{rc}\hat{\phi}$$
$$\vec{B} = \nabla\times\vec{A}$$
$$= \begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\hat{\theta}$$
$$= \frac{\mu_0 b^2}{4c}\left[\frac{1}{r\sin\theta}\frac{\dot{I}}{r}(2\sin\theta\cos\theta)\hat{r} - \frac{1}{r}\sin\theta\left(-\frac{\ddot{I}}{c}\right)\hat{\theta}\right]$$
$$\vec{B}_{\text{rad}} = \frac{\mu_0 b^2}{4c^2}\ddot{I}\frac{\sin\theta}{r}\hat{\theta}$$

The Poynting vector is

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0}\left(\vec{E}_{\text{rad}}\times\vec{B}_{\text{rad}}\right)$$
$$= \frac{1}{\mu_0 c}\left(\frac{\mu_0 b^2}{4c^2}\ddot{I}\frac{\sin\theta}{r}\right)^2\left(-\hat{\phi}\times\hat{\theta}\right)$$
$$= \frac{\mu_0}{16c^3}\left(b^2\ddot{I}\right)^2\frac{\sin^2\theta}{r^2}\hat{r}$$

The intensity is

$$\left\langle \vec{S}_{\text{rad}} \right\rangle = \frac{\mu_0 b^4 I_0^2}{16c^3} \frac{\sin^2\theta}{r^2} \int_0^T \ddot{I}^2 \, dt_0 \hat{r}$$

$$= \frac{\mu_0 b^4 I_0^2}{16c^3} \frac{\sin^2\theta}{r^2} \int_0^T \omega^4 \cos^2(\omega t_0) \, dt_0 \hat{r}$$

$$= \frac{\mu_0 b^4 I_0^2 \omega^4}{32c^3} \frac{\sin^2\theta}{r^2} \hat{r}$$

The total power radiated is

$$P = \int \left\langle \vec{S}_{\text{rad}} \right\rangle \cdot d\vec{a}$$

$$= \frac{\mu_0 b^4 I_0^2 \omega^4}{32c^3} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta d\phi$$

$$= \frac{\mu_0 b^4 I_0^2 \omega^4}{32c^3} \cdot \frac{4}{3} \cdot 2\pi$$

$$= \frac{\mu_0 \pi b^4 I_0^2 \omega^4}{12c^3}$$

Let

$$m_0 = \pi b^2 I_0$$

We obtain

$$P = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

EXAMPLES:

1. In the Rutherfold model of hydrogen atom, the electron is circulating around the nucleus. Thus, the atom can be seen as a rotating electric dipole:

   $$\vec{r}(t) = r_0(\cos\omega t \, \hat{x} + \sin\omega t \, \hat{y})$$

   and

   $$\vec{p}(t) = -e\vec{r}(t)$$

   Find the time for the radius of the electron to shrink from a_0 to zero.

   ANSWER:

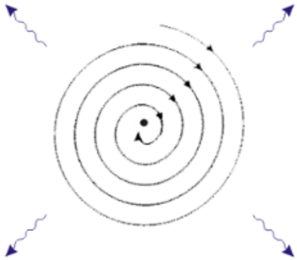
   According to the Larmor formula,

   $$P = \frac{\mu_0 e^2 a^2}{6\pi c}$$

   If the radius of the electron orbit is r, then its energy is

   $$E = \frac{1}{2} m v^2 - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{e^2}{8\pi\epsilon_0 r}$$

When the electron radiates and loses energy, the radius shrinks,



the change of energy is,

$$dE = \frac{e^2}{8\pi\epsilon_0 r^2} dr, \qquad (dr < 0)$$

Thus,

$$Pdt = |dE|$$
$$\Rightarrow \frac{\mu_0 e^2 a^2}{6\pi c} dt = -\frac{e^2}{8\pi\epsilon_0 r^2} dr$$
$$\Rightarrow dt = -\frac{3c}{4\mu_0\epsilon_0 a^2 r^2} dr = -\frac{3c^3}{4a^2 r^2} dr$$

Since

$$a = \omega^2 r = \frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 m r^2} \Rightarrow a^2 r^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{m^2 r^2}$$

this leads to,

$$dt = -\frac{3c^3}{4} \frac{1}{\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{m^2 r^2}} dr = -\frac{3c^3}{4c^4 r_e^2} r^2 dr = -\frac{3}{4c r_e^2} r^2 dr$$

where

$$r_e = \frac{e^2}{4\pi\epsilon_0 m c^2} = \alpha^2 a_0, \qquad \alpha \approx 1/137$$

The time for the radius of the electron to shrink from a_0 to zero.

$$\tau = -\frac{3}{4c\alpha^4 a_0^2} \int_{a_0}^{0} r^2\, dr$$
$$= -\frac{3}{4c\alpha^4 a_0^2} \cdot \left(-\frac{a_0^3}{3}\right)$$
$$= \frac{a_0}{4c\alpha^4}$$
$$\approx 1.31 \times 10^{-11}\ \mathrm{s}$$

That is, according to classical theory, the Rutherford atom should collapse immediately because of the radiation energy loss.